Unruh effect via Bogoliubov coefficient approach

Ashmita Das*
Department of Physics, School of Engineering and Sciences, SRM University AP

Contents

I. Some basic recapitulations

We define the vacuum state $|0\rangle$ which is annihilated to zero while operated by an annihilation operator such as: $a_{\mathbf{p}}|0\rangle = 0$.

- A form of Hamiltonian is given by as below:

$$H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[a_{\mathbf{p}}\,a^{\dagger}_{\mathbf{p}} + a^{\dagger}_{\mathbf{p}}\,a_{\mathbf{p}}\right] \tag{1}$$

Derive the ground state energy eigen value corresponding to this Hamiltonian. You can start from, $H|0\rangle \equiv E_0|0\rangle$

An important fundamental aspect of QFT:

$$H|0\rangle \;\equiv\; E_0|0\rangle \;=\; \left[\int d^3p\,\frac{1}{2}\,\omega_{\vec{p}}\,\delta^{(3)}(0)\right]|0\rangle \;=\; \infty\,|0\rangle. \tag{2}$$

The subject of quantum field theory is full with infinities. Each tells us something important, usually that we are doing something wrong, or asking the wrong question. Let's take some time to explore where this infinity comes from and how we should deal with it. In fact there are two different ∞'s hanging around in the expression (2). The first arises because space is infinitely large. (Infinities of this type are often referred to as *infra-red divergences* although in this case the ∞ is so simple that it barely deserves this name). To extract out this infinity, let us consider putting the theory in a box with sides of length L. We impose periodic boundary conditions on the field. Then, taking the limit where $L \to \infty$, we get

$$(2\pi)^3\,\delta^{(3)}(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x\; e^{\,i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=0} = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x = V \tag{3}$$

where V is the volume of the box. So the $\delta(0)$ divergence arises because we are computing the total energy, rather than the energy density $\mathcal{E}_0$. To find $\mathcal{E}_0$ we can simply divide by the volume,

$$\mathcal{E}_0 \;=\; \frac{E_0}{V} \;=\; \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\,\omega_{\vec{p}}. \tag{4}$$

which is still infinite. We recognize it as the sum of ground state energies for each harmonic oscillator. But $\mathcal{E}_0\tau 0\infty$ due to the $|\mathbf{p}| \to \infty$ limit of the integral. This is a high frequency — or short distance infinity, known as an ultra-violet divergence. This divergence arises because we have assumed that our theory is valid to arbitrarily short distance scales, corresponding to arbitrarily high energies. This is clearly absurd. The integral should have a cut-off at high momentum in order to reflect the fact that our theory is likely to break down in some way.

* ashmita.d@srmap.edu.in

II. Understanding field decomposition in QFT in two dimensions

The massless scalar field in two dimensions, $\hat{\Phi}(t,z)$, satisfies,

$$\left(\partial^2/\partial t^2 - \partial^2/\partial z^2\right)\hat{\Phi}(t,z) = 0 \tag{5}$$

This field can be expanded as,

$$\hat{\Phi}(t,z) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}}\left(\hat{b}_{-k}e^{-ik(t-z)} + \hat{b}_{+k}e^{-ik(t+z)} + \hat{b}^\dagger_{-k}e^{ik(t-z)} + \hat{b}^\dagger_{+k}e^{ik(t+z)}\right). \tag{6}$$

In the above equation note that it has been decomposed as left and right moving modes. It can be perceived that $e^{ik(t+z)}$, $e^{-ik(t+z)}$ are the left moving modes. Now let us consider,

$$U = t - z, \quad V = t + z,$$

Thus we can write

$$\hat{\Phi}(t,z) = \hat{\Phi}_-(U) + \hat{\Phi}_+(V), \tag{7}$$

where

$$\hat{\Phi}_+(V) = \int_0^\infty dk\left[\hat{b}_{+k}f_k(V) + \hat{b}^\dagger_{+k}f^*_k(V)\right] \tag{8}$$

with

$$f_k(V) = (4\pi k)^{-1/2}e^{-ikV}, \tag{9}$$

and similarly for $\hat{\Phi}_-(U)$.

- Verify the Eqs. (7), (8) and (9) using $U\,V$ as a function of t, z.

Since the left and right-moving sectors of the field, $\hat{\Phi}_+(V)$ and $\hat{\Phi}_-(U)$, do not interact with one another, we discuss only the left-moving sector $\hat{\Phi}_+(V)$. (Thus, we will discuss the Unruh effect for the theory consisting only of the left-moving sector.) The Minkowski vacuum state $|0_M\rangle$ is defined by $\hat{b}_{+k}|0_M\rangle = 0$ for all k.

III. Scalar field equation in Rindler coordinate and derivation of Bogoliubov coefficients

Using the metric in the right Rindler wedge given by,

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) - dx^2 - dy^2\,, \tag{10}$$

one finds a field equation of the same form as Eq. (5):

$$(\partial^2/\partial\tau^2 - \partial^2/\partial\xi^2)\hat{\Phi} = 0. \tag{11}$$

(This is a result of the conformal invariance of the massless scalar field theory in two dimensions.)

- Verify that scalar field equation in the Rindler spacetime turns out to be as Eq.(11).

Recall the general trajectory of the observer in RRW as:

$$t = \frac{1}{a}\,e^{a\xi}\sinh a\tau\,, \qquad z = \frac{1}{a}\,e^{a\xi}\cosh a\tau\ . \tag{12}$$

Note: for the proper frame condition of the observer i.e. ($\xi = 0$), we get the usual Rindler transformations as stated in the previous lectures. The solutions to this differential equation can be classified again into the left and righ moving modes which depend only on $v = \tau + \xi$ and $u = \tau - \xi$, respectively. These variables are related to U and V as follows:

$$U = t - z = -a^{-1}e^{-au}, \tag{13}$$

$$V = t + z = a^{-1}e^{av}. \tag{14}$$

- Verify the Eqs. (13) and (14).

The Lagrangian density is invariant under the coordinate transformation $(t,z)\mapsto(\tau,\xi)$. As a result, by going through the quantization procedure demonstrated in the previous lecture, one finds exactly the same theory as in the whole of Minkowski spacetime with (t,z) replaced by (τ,ξ). Thus, we have, for $0<V$,

$$\hat{\Phi}_+(V)=\int_0^\infty dw\left[\hat{a}^R_{+\omega}g_\omega(v)+\hat{a}^{R\dagger}_{+\omega}g^*_\omega(v)\right], \tag{15}$$

where following Eq. (9) we are able to write,

$$g_\omega(v)=(4\pi\omega)^{-1/2}e^{-i\omega v}, \tag{16}$$

and where

$$[\hat{a}^R_{+\omega},\hat{a}^{R\dagger}_{+\omega'}]=\delta(\omega-\omega') \tag{17}$$

with all other commutators vanishing. Notice that the functions $g_\omega(v)$ are eigenfunctions of the boost generator $\partial/\partial\tau$. The field $\hat{\Phi}_+(V)$ can be expressed in the left Rindler wedge with the condition $V<0<U$, by using the left Rindler coordinates $(\bar{\tau},\xi)$ defined as,

$$t=\frac{1}{a}\,e^{a\bar{\xi}}\sinh a\bar{\tau}\,,\qquad\qquad z=-\frac{1}{a}\,e^{a\bar{\xi}}\cosh a\bar{\tau}\ . \tag{18}$$

Defining $\bar{v}=\bar{\tau}-\xi$, one obtains Eqs. (15) to (17) with v replaced by $\bar{v}$ and with the annihilation and creation operators $\hat{a}^R_{+\omega}$ and $\hat{a}^{R\dagger}_{+\omega}$ replaced by a new set of operators $\hat{a}^L_{+\omega}$ and $\hat{a}^{L\dagger}_{+\omega}$.

- Verify that the variable $\bar{v}$ is related to V by,

$$V=-a^{-1}e^{-a\bar{v}}. \tag{19}$$

Also note that

$$\hat{\Phi}_+(V)=\int_0^\infty d\omega\left\{\theta(V)\left[\hat{a}^R_{+\omega}g_\omega(v)+\hat{a}^{R\dagger}_{+\omega}g^*_\omega(v)\right]+\theta(-V)\left[\hat{a}^L_{+\omega}g_\omega(\bar{v})+\hat{a}^{L\dagger}_{+\omega}g^*_\omega(\bar{v})\right]\right\}, \tag{20}$$

The static vacuum state in the left and right Rindler wedges, the Rindler vacuum state $|0_R\rangle$, is defined by $\hat{a}^R_{+\omega}|0_R\rangle=\hat{a}^L_{+\omega}|0_R\rangle=0$ for all ω.
To understand the Unruh effect we need to find the Bogolubov coefficients $\alpha^R_{\omega k}$, $\beta^R_{\omega k}$, $\alpha^L_{\omega k}$ and $\beta^L_{\omega k}$, where

$$\theta(V)g_\omega(v)=\int_0^\infty\frac{dk}{\sqrt{4\pi k}}\left(\alpha^R_{\omega k}e^{-ikV}+\beta^R_{\omega k}e^{ikV}\right), \tag{21}$$

$$\theta(-V)g_\omega(\bar{v})=\int_0^\infty\frac{dk}{\sqrt{4\pi k}}\left(\alpha^L_{\omega k}e^{-ikV}+\beta^L_{\omega k}e^{ikV}\right). \tag{22}$$

Here, $\theta(x)=0$ if $x<0$ and $\theta(x)=1$ if $x>0$, i.e. θ is the Heaviside function. To find $\alpha^R_{\omega k}$ we multiply Eq. (2.53) by $e^{ikV}/2\pi$, $k>0$, and integrate over V. Thus, we find:

$$\alpha^R_{\omega k}=\sqrt{4\pi k}\int_0^\infty\frac{dV}{2\pi}g_\omega(V)e^{ikV}$$

$$=\sqrt{\frac{k}{\omega}}\int_0^\infty\frac{dV}{2\pi}(aV)^{-i\omega/a}e^{ikV}.$$

We introduce a cut-off for this integral for large V by letting $V\to V+i\varepsilon$, $\varepsilon\to0+$. Then, changing the integration path to the positive imaginary axis by letting $V=ix/k$, we find

$$\alpha^R_{\omega k}=\frac{ie^{\pi\omega/2a}}{\sqrt{\omega k}}\left(\frac{a}{k}\right)^{-i\omega/a}\int_0^\infty\frac{dx}{2\pi}x^{-i\omega/a}e^{-x}dx$$

$$= \frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a). \tag{23}$$

To find the coefficients $\beta^R_{\omega k}$ we replace e^{ikV} in Eq. (III) by e^{-ikV}. Then, the appropriate substitution is $V = -ix/k$. As a result we obtain:

$$\beta^R_{\omega k} = -\frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{-i\omega/a} \Gamma(1 - i\omega/a). \tag{24}$$

A similar calculation leads to

$$\alpha^L_{\omega k} = -\frac{ie^{\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{i\omega/a} \Gamma(1 + i\omega/a), \tag{25}$$

$$\beta^L_{\omega k} = \frac{ie^{-\pi\omega/2a}}{2\pi\sqrt{\omega k}} \left(\frac{a}{k}\right)^{i\omega/a} \Gamma(1 + i\omega/a). \tag{26}$$

We find that these coefficients obey the following relations crucial to the derivation of the Unruh effect:

$$\beta^L_{\omega k} = -e^{-\pi\omega/a}\alpha^{R*}_{\omega k}, \quad \beta^R_{\omega k} = -e^{-\pi\omega/a}\alpha^{L*}_{\omega k}. \tag{27}$$

By substituting these relations in Eqs. (21) and (22), we find that the following functions are linear combinations of positive frequency modes e^{-ikV} in Minkowski spacetime:

$$G_\omega(V) = \theta(V)g_\omega(v) + \theta(-V)e^{-\pi\omega/a}g^*_\omega(\overline{v}), \tag{28}$$

$$\bar{G}_\omega(V) = \theta(-V)g_\omega(\overline{v}) + \theta(V)e^{-\pi\omega/a}g^*_\omega(v). \tag{29}$$

One can show that these functions are purely positive frequency solutions in Minkowski spacetime by analyticity argument as well: since a positive frequency solution is analytic in the lower half plane on the complex V-plane, the solution $g_\omega(v) = (4\pi\omega)^{-1/2}(V)^{-i\omega/a}$, $V > 0$, should be continued to the negative real line avoiding the singularity at $V = 0$ around a small circle in the lower half-plane, thus leading to $(4\pi\omega)^{-1/2}e^{-\pi\omega/a}(-V)^{-i\omega/a}$ for $V < 0$. This was the original argument by Unruh (1976).
Eqs. (28) and (29) can be inverted as

$$\theta(V)g_\omega(v) \propto G_\omega(V) - e^{-\pi\omega/a}\bar{G}^*_\omega(V), \tag{30}$$

$$\theta(-V)g_\omega(\overline{v}) \propto \bar{G}_\omega(V) - e^{-\pi\omega/a}G^*_\omega(V). \tag{31}$$

By substituting these equations in

$$\hat{\Phi}_+(V) = \int_0^\infty d\omega \left\{\theta(V)\left[\hat{a}^R_{+\omega}g_\omega(v) + \hat{a}^{R\dagger}_{+\omega}g^*_\omega(v)\right] + \theta(-V)\left[\hat{a}^L_{+\omega}g_\omega(\overline{v}) + \hat{a}^{L\dagger}_{+\omega}g^*_\omega(\overline{v})\right]\right\}, \tag{32}$$

we find that the integrand here is proportional to

$$G_\omega(V)\left[\hat{a}^R_{+\omega} - e^{-\pi\omega/a}\hat{a}^{L\dagger}_{+\omega}\right] + \overline{G}_\omega(V)\left[\hat{a}^L_{+\omega} - e^{-\pi\omega/a}\hat{a}^{R\dagger}_{+\omega}\right] + \text{H.c.} \tag{33}$$

Since the functions $G_\omega(V)$ and $\overline{G}_\omega(V)$ are positive frequency solutions (with respect to the usual time translation) in Minkowski spacetime, the operators $\hat{a}^R_{+\omega} - e^{-\pi\omega/a}\hat{a}^{L\dagger}_{+\omega}$ and $\hat{a}^L_{+\omega} - e^{-\pi\omega/a}\hat{a}^{R\dagger}_{+\omega}$ annihilate the Minkowski vacuum state $|0_M\rangle$. Thus,

$$\left(\hat{a}^R_{+\omega} - e^{-\pi\omega/a}\hat{a}^{L\dagger}_{+\omega}\right)|0_M\rangle = 0, \tag{34}$$

$$\left(\hat{a}^L_{+\omega} - e^{-\pi\omega/a}\hat{a}^{R\dagger}_{+\omega}\right)|0_M\rangle = 0. \tag{35}$$

These relations uniquely determine the Minkowski vacuum $|0_M\rangle$ as we explain below.
To explain how the state $|0_M\rangle$ is formally expressed in the Fock space on the Rindler vacuum state $|0_R\rangle$ and to

show that the state $|0_M\rangle$ is a thermal state when it is probed only in the right (or left) Rindler wedge, we use the approximation where the Rindler energy levels ω are discrete. This writing ω_i instead ω and get the commutations as:

$$\left[\hat{a}^R_{\omega_i},\, \hat{a}^{R\dagger}_{\omega_j}\right] = \left[\hat{a}^L_{\omega_i},\, \hat{a}^{L\dagger}_{\omega_j}\right] = \delta_{ij} \tag{36}$$

with all other commutators among $\hat{a}^R_{+\omega_i}$, $\hat{a}^L_{+\omega_i}$ and their Hermitian conjugates vanishing. Using the discrete version of Eqs. (34) and the commutators (36), we find

$$\langle 0_M|\hat{a}^{R\dagger}_{+\omega_i}\hat{a}^R_{+\omega_i}|0_M\rangle = e^{-2\pi\omega_i/a}\langle 0_M|\hat{a}^L_{+\omega_i}\hat{a}^{L\dagger}_{+\omega_i}|0_M\rangle + e^{-2\pi\omega_i/a}. \tag{37}$$

The same relation with $\hat{a}^R_{+\omega_i}$ and $\hat{a}^{R\dagger}_{+\omega_i}$ replaced by $\hat{a}^L_{+\omega_i}$ and $\hat{a}^{L\dagger}_{+\omega_i}$, respectively and vice versa, can be found using Eq. (35). By solving these two relations as simultaneous equations, we find

$$\langle 0_M|\hat{a}^{R\dagger}_{+\omega_i}\hat{a}^R_{+\omega_i}|0_M\rangle = \langle 0_M|\hat{a}^{L\dagger}_{+\omega_i}\hat{a}^L_{+\omega_i}|0_M\rangle = \frac{1}{e^{2\pi\omega_i/a} - 1}. \tag{38}$$

Hence, the expectation value of the Rindler particle number is that of a Bose-Einstein particle in a thermal bath of temperature $T = a/2\pi$. This indicates that the Minkowski vacuum can be expressed as a thermal state in the Rindler wedge.